Reprojection using a parallel backprojector

Carl R. Crawford[a)]
Elscint Ltd., P. O. Box 5258, Haifa, Israel

(Received 21 November 1985; accepted for publication 12 March 1986)

Reprojection is the process by which projections are produced from an image, such that if these projections are filtered and backprojected, they yield the original image. In computed tomography, applications of reprojection include its use in algorithms for iterative beam hardening correction, metal artifact removal, and streak suppression. However, because of the computational expense of reprojection, algorithms that employ this process have never been widely used. A method will be presented that enables an unmodified backprojector to be used as a reprojector. Because backprojectors are designed to exploit the parallelism in the backprojection algorithm, the time required to obtain reprojections is significantly reduced.

I. INTRODUCTION

In a standard computed tomographic (CT) scanner, the outputs of suitable detectors are processed to produce estimates of line integrals through a two-dimensional function representing a characteristic of the object under examination. The line integrals for a given rotational position of the gantry are called the projection of the object. In a typical system, projections are collected for a large number of rotational positions.

Generally, an image is reconstructed by filtering the projections with a one-dimensional function and then backprojecting these filtered projections along the projection paths. The result of this backprojection operation is an image that closely represents the object function.

Consider a new situation in which one begins with an image and produces projections, such that if these projections were filtered and backprojected, they yield the original image. This process is called reprojection. With this definition, projection represents the interactions of a form of radiation with an object while reprojection is a mathematical process.

In x-ray CT, the best known application of reprojection is its use in iterative beam hardening correction algorithms.[1] However, other uses of reprojection can be found in streak suppression algorithms[2] and algorithms that are used to remove artifacts caused by the presence of metal clips in CT images.[3]

The readily available reprojection algorithms are very computationally expensive. This computational expense has been one of the major reasons why algorithms that utilize reprojection have never been widely used. Thus there exists the need for faster reprojection methods so that algorithms based on reprojection can be implemented in conventional systems and execute in times that satisfy the clinical need.

The slowness of reprojection systems and an attempted solution are highlighted in the paper by Peters[4] in which a method is shown that uses a modified backprojector to obtain reprojections. The modifications reverse the normal data flow so that reprojections are generated at the normal input of the unit. The problem with this approach is that the modifications radically change the hardware of a typical backprojector, and thus the system is not readily applicable to existing devices. In addition, the resulting reprojections are of poor quality and require complex corrections in order to be useful.

Another approach used to reduce the processing time in systems that utilize reprojection is to reproject data along parallel paths even though the image to be reprojected was reconstructed using fan beam projections or to reproject at fewer paths than the original data set.[5] However, even with these simplifications, reprojection is still a computationally expensive process.

A review will first be presented of the mathematics of parallel backprojection along with a description of an efficient implementation of the algorithm. A discussion of the mathematics of reprojection will then be presented. Then a method will be presented that enables an unmodified backprojector to be used as a reprojector. Because normal data paths of the backprojector are used and because backprojectors are already present in most, if not all CT scanners, the time required to obtain reprojections is significantly reduced. Finally, a discussion of the new system and conclusions will be stated.

II. PARALLEL BACKPROJECTION

Let $f(x, y)$ be a function of an object to be reconstructed from its parallel projections. Consider the ts-coordinate system obtained when the xy-coordinate system is rotated by θ. The two coordinate systems are related as follows:

$$t = x\cos\theta + y\sin\theta, \tag{1}$$

$$s = y\cos\theta - x\sin\theta. \tag{2}$$

Let $f^*(t,s)$ be the object function in the ts system. A parallel projection of $f(x, y)$ in the θ direction, $p(\theta,t)$, is given by

$$p(\theta,t) = \int_{-\infty}^{\infty} f^*(t,s)ds. \tag{3}$$

$S(\theta,\omega)$, the Fourier transform of $p(\theta,t)$, is given by

$$S(\theta,\omega) = \int_{-\infty}^{\infty} p(\theta,t)e^{-j2\pi\omega t}\,dt. \tag{4}$$

Define the filtered projection, $q(\theta,t)$, as follows:

$$q(\theta,t) = \int_{-\infty}^{\infty} S(\theta,\omega)K(\omega)e^{j2\pi\omega t}\,d\omega, \tag{5}$$

where $K(\omega)$ is the reconstruction kernel given by the pro-

 0094-2405/86/040480-04$01.20

duct of $|\omega|$ and $G(\omega)$. $G(\omega)$ is typically a low-pass filter used to tailor the resolution and noise characteristics of the reconstructed image.[6] It can be shown[7] that the original function, $f(x,y)$, can be reconstructed using the following integral equation:

$$f(x,y) = \int_0^{\pi} q(\theta, x\cos\theta + y\sin\theta)d\theta. \quad (6)$$

In reality, only a finite number of projections are collected and each projection is sampled at a finite number of points. The sampled projection, $P(l,i)$, can be related to the continuous projection by

$$P(l,i) = p(l\Delta_\theta, i\Delta_t), \quad (7)$$

where

$$l = 0,1,...,\text{NVIEW} - 1, \quad (8)$$

$$i = -\text{NDET}/2 + 1,...,0,...,\text{NDET}/2, \quad (9)$$

$$\Delta_\theta = \pi/\text{NVIEW}, \quad (10)$$

$$\Delta_t = \text{FOV}/\text{NDET}. \quad (11)$$

NVIEW is the number of projections (or views), NDET is the number of samples (or detectors) per projection, and FOV is the diameter of the field-of-view in which the object is contained. Let $Q(l,i)$ be a sampled version of $q(\theta,t)$ obtained by convolving $P(l,i)$ with a sampled version of the inverse Fourier transform of $K(\omega)$. Then the reconstruction integral given in Eq. (6) can then be approximated by[8]

$$f(x,y) \simeq \Delta_\theta \sum_{l=0}^{\text{NVIEW}-1} Q(l, x\cos\theta + y\sin\theta). \quad (12)$$

In Eq. (12), values of Q will be required at locations where the function is not sampled. Generally, it is assumed that these values are obtained by linearly interpolating neighboring samples.

Using Eq. (12), it is possible to reconstruct $f(x,y)$ at any value of (x,y). In practice, many values of $f(x,y)$, say an $N \times N$ array, are required. Let $F(m,n)$ be the sampled version of the continuous image function such that

$$F(m,n) = f(m\Delta_f, n\Delta_f), \quad (13)$$

where

$$m,n = -N/2 + 1,...,0,...,N/2, \quad (14)$$

$$\Delta_f = \text{FOV}/N. \quad (15)$$

Then, in practice, a simple reconstruction algorithm based on Eq. (12) could be invoked N^2 times to reconstruct $F(m,n)$. One limitation with this algorithm is that it cannot be invoked until all of the projections are collected. Thus a pipeline architecture is precluded. Also, it is computationally expensive to have trigonometric operations in the innermost loop of the algorithm.

Let $t(x,y)$ be the value of t for a given value of (x,y). From Eq. (1) it can be seen that

$$t(x + \Delta_f, y) = t(x,y) + \Delta_f\cos\theta, \quad (16)$$

$$t(x, y + \Delta_f) = t(x,y) + \Delta_f\sin\theta. \quad (17)$$

Using Eqs. (12), (16), and (17), an efficient reconstruction algorithm for an $N \times N$ array can be developed as shown in Fig. 1. [Note that the routine reconstructs $f(x,y)/\Delta_\theta$ instead of $f(x,y)$ and the subroutine INTERP linearly interpolates a value of $Q(l,i)$ at t.]

```
Zero all locations of F(m,n)
DO ℓ=0,NVIEW-1
   θ = ℓ Δθ
   Δx = Δf cosθ
   Δy = Δf sinθ
   t0 = -(N/2 - 1) Δx - (N/2 - 1) Δy
   DO m=-N/2+1,N/2
      t = t0
      DO n=-N/2+1,N/2
         F(m,n) = F(m,n) + INTERP(Q,t)
         t = t + Δx
      END DO
      t0 = t0 + Δy
   END DO
END DO
```

FIG. 1. An efficient algorithm for the reconstruction of an image of size $N \times N$.

The algorithm of Fig. 1 forms the basis of typical hardware implementations of backprojection in which the underlined code is usually implemented in hardware. The rest of the steps are performed in a flexible coprocessor. The role of the coprocessor is to supply the backprojector with the filtered projection data, the size of the image, t_0, $\Delta_f\cos\theta$, and $\Delta_f\sin\theta$.

III. MATHEMATICS OF REPROJECTION

Let $f(x,y)$ be the reconstruction of the cross section of an object. The parallel reprojection of the reconstruction in the θ direction, $r(\theta,t)$, is given by

$$r(\theta,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(t - x\cos\theta - y\sin\theta)dx\,dy. \quad (18)$$

A sample of $r(\theta,t)$ represents the line integral of $f(x,y)$ along the path given in Eq. (1). As presented in the previous section, a discrete version of $f(x,y)$, $F(m,n)$, is typically reconstucted. An approximate value of the reprojection can be found by substituting Eq. (13) into Eq. (18) and replacing the integrals with summations:

$$r(\theta,t) \simeq \sum_{m=-N/2+1}^{N/2}\sum_{n=-N/2+1}^{N/2} w(m,n;\theta,t)F(m,n), \quad (19)$$

where the function $w(m,n;\theta,t)$ is the distance through a pixel located at (m,n) for the line given by the parametric relationship (θ,t). Figure 2 shows an example in which the values of $w(m,n;\theta,t)$ can be discerned.

For now, consider only integration paths (θ,t) which satisfy the following:

$$|\sin\theta| < 1/\sqrt{2}. \quad (20)$$

Further, consider the intersection of the path (θ,t) with the mth row in the image and assume that the ray intersects this row between pixels (m,n) and $(m,n+1)$ as shown in Fig. 3. Given m, θ, and t, the value of n can be determined by solving for x in Eq. (1) and then using Eq. (13):

$$n = \text{INT}(n'), \quad (21)$$

where

$$n' = (t - m\,\Delta_f\sin\theta)/(\Delta_f\cos\theta), \quad (22)$$

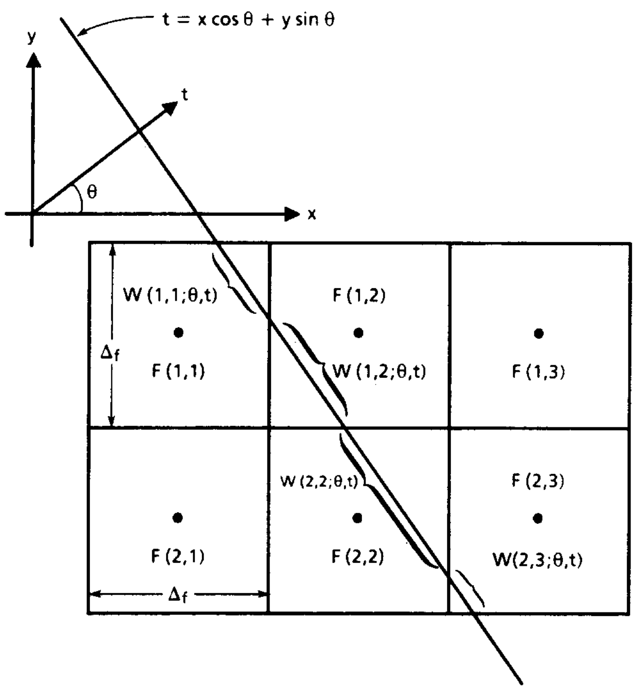

FIG. 2. An example in which the weights, $w(m,n;\theta,t)$, of Eq. (19) can be discerned.

and INT(x) is the greatest integer less than or equal to x. Because of Eq. (20), only $w(m,n;\theta,t)$ and $w(m,n+1;\theta,t)$ can be nonzero. Thus, the contribution of the mth row to the reprojection, $r(\theta,t;m)$, is given by the weighted sum of the pixel between which the ray crosses:

$$r(\theta,t;m) \simeq w(m,n;\theta,t)F(m,n) + w(m,n+1;\theta,t)F(m,n+1). \tag{23}$$

Joseph[9] showed that a better approximation to Eq. (18) is obtained if the weights in Eq. (23) are those that reduce to linear interpolation between the pixel values instead of the geometric path lengths depicted in Fig. 2. With linear interpolation, the values of the weight functions reduce to the following:

$$w(m,n;\theta,t) = \Delta_f\,(n'+1-n)/|\cos\theta|, \tag{24}$$

$$w(m,n+1;\theta,t) = \Delta_f\,(n'-n)/|\cos\theta|. \tag{25}$$

Figure 4 shows an algorithm, using Eqs. (23), (24), and (25), to calculate, within a scale factor, a single ray of a reprojection. In the figure, the notation F_m corresponds to the mth row of $F(m,n)$.

For a reprojection system to be of use, all the rays comprising a reprojection are needed. Assume that the reprojections have to be calculated along the original projection paths as defined in Eq. (7). Then the discrete samples of the reprojection are given by

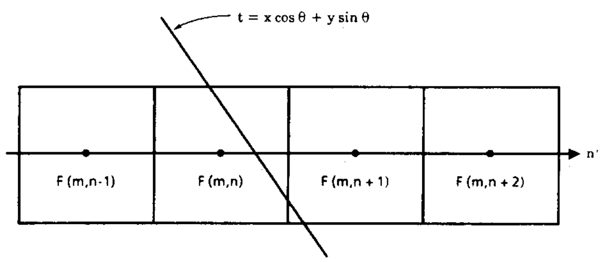

FIG. 3. Intersection of a ray of a reprojection with the mth row of an image.

```
r(θ,t) = 0
DO m=-N/2+1,N/2
   n' = (t - m Δf sinθ) / (Δf cosθ)
   R(θ,i) = R(θ,i) + INTERP(Fm(n),n')
END DO
```

FIG. 4. An algorithm for reprojection along a single ray.

$$R(\theta,i) = r[\theta,t(i)], \tag{26}$$

where i is defined in Eq. (9) and

$$t(i) = i\,\Delta_t. \tag{27}$$

Again, one could simply invoke the algorithm of Fig. 4 once for each of the samples in the reprojection. However, this procedure does not exploit the common intermediate calculations in the steps by which several reprojection samples are computed.

Let $n'(i)$ be the value of n' for a particular value of $t(i)$. It can be seen from Eqs. (22) and (27) that $n'(i)$ and the value of n' for the next ray, $n'(i+1)$, are related as follows:

$$n'(i+1) = n'(i) + \Delta_t', \tag{28}$$

where

$$\Delta_t' = \Delta_t/(\Delta_f\cos\theta). \tag{29}$$

Using Eq. (28) the algorithm shown in Fig. 4 can be extended to reproject efficiently multiple samples in a reprojection as shown in Fig. 5. Note that algorithm was extended with an extra redundant loop.

If the underlined lines of this algorithm are compared to the parallel backprojection algorithm shown in Fig. 1, the similarities are striking. Notice that in backprojection, filtered projections and the image are analogous to rows of an image and the computed reprojection, respectively. Thus, reprojection can be implemented by "backprojecting" the N rows of the image as "filtered projections," each containing N samples, resulting in a reconstructed "image" of size NDET by 1. The image is the reprojection. The coprocessor of the backprojector can be used to generate Δ_t' and n_0, thus allowing the backprojector to generate reprojections.

The algorithm shown in Fig. 5 was derived with the assumption presented in Eq. (20). This assumption was need-

```
Zero the samples of the reprojection
DO m=-N/2+1,N/2
   Δt' = Δt / (Δf cosθ)
   n0= -(NDET/2 - 1) Δt' - m tanθ
   DO j=1,1
      n' = n0
      DO i=-NDET/2+1,NDET/2
         R(θ,i) = R(θ,i) + INTERP(Fm(n),n')
         n'= n' + Δt'
      END DO
   END DO
END DO
```

FIG. 5. An efficient reprojection algorithm for multiple samples with an extra loop so that it matches the algorithm shown in Fig. 1.

ed so that a ray crosses, at most, two samples in a row. For values of θ that do not satisfy Eq. (20), the role of the rows and the columns in an image have to be switched. In this latter case, the columns rather than the rows of the image should be backprojected.

IV. DISCUSSION

The reprojection algorithm using a parallel backprojector was implemented on an Elscint 2002 CT scanner. It was found that the backprojector was able to reproject a data set comparable in size to the original projection data in approximately the same time required to reconstruct an original image. The reprojections were used as part of an iterative beam hardening correction algorithm.

The description of a backprojector in Sec. II applies to a parallel backprojection unit. However, it can be seen that, with the proper choice of constants that describe the geometry of the fan beam, a fan beam backprojector can also be used as a parallel backprojector. However, the algorithm shown in the previous section cannot be extended to allow fan beam reprojection.

V. CONCLUSIONS

A method that allows an unmodified backprojector to be used as a reprojector has been shown. The advantage of this configuration is that it takes advantage of the fact that backprojectors are already present in most, if not all CT scanners, and are designed to exploit the parallelism in the backprojection algorithm. The time required to obtain reprojections is sufficiently reduced so that applications requiring reprojection can be implemented in conventional systems and execute in times that satisfy the clinical need. The parallel backprojector on an Elscint 2002 CT scanner was used to obtain reprojections in a time comparable to that required for the original backprojection.

ACKNOWLEDGMENTS

The author is grateful to Dr. J. Colsher, Dr. K. King, Dr. A. Lonn, and Dr. N. Pelc for many useful discussions on this subject and for editing the manuscript. The author would also like to thank G. Greuel for preparing the line drawings.

[a] Present address: Applied Science Laboratory, Medical Systems Group, General Electric Company, P. O. Box 414, W-875, Milwaukee, WI 53201.

[1] P. M. Joseph and R. D. Spital, J. Comput. Assist. Tomogr. **2**, 100 (1978).
[2] G. Henrich, Comput. Tomogr. **4**, 67 (1980).
[3] G. H. Glover and N. J. Pelc, Med. Phys. **8**, 799 (1981).
[4] T. M. Peters, IEEE Trans. Nucl. Sci. **28**, 3641 (1981).
[5] J. F. Moore, U. S. Patent No. 4 222 104, 1980.
[6] A. Rosenfeld and A. C. Kak, *Digital Picture Processing—Second Edition* (Academic, New York, 1982).
[7] R. N. Bracewell and A. C. Riddle, Astrophys. J. **150**, 427 (1967).
[8] C. R. Crawford and A. C. Kak, Appl. Opt. **18**, 3704 (1979).
[9] P. M. Joseph, IEEE Trans. Med. Imaging **1**, 192 (1982).