Reconstruction for fan beam with an angular-dependent displaced center-of-rotation

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A convolutional backprojection algorithm is derived for a fan beam geometry that has an angular-dependent displacement in its center-of-rotation from the midline of the fan beam. In both x-ray computed tomography and single photon emission computed tomography, misalignment can occur when the mechanical center-of-rotation is not colinear with midline of the fan beam. In some cases the shift in the center-of-rotation is constant for every angle, whereas, in other cases it varies with angular position. Standard reconstruction algorithms, which directly filter and backproject the fan beam data without rebinning into parallel beam geometry, have been derived for a geometry having its center-of-rotation at the midline of the fan beam. However, in the case of any misalignment of the center-of-rotation, if these conventional reconstruction algorithms are used to reconstruct the fan beam projections, structured artifacts and a loss of resolution will result. Simulations are performed that illustrate these artifacts and demonstrate how the new algorithm corrects for this misalignment. A method for estimating the parameters of the fan beam geometry, including the angular-dependent shift in the center-of-rotation, is also described.

I. INTRODUCTION

Fan beam reconstruction algorithms were first derived based on the assumption that the mechanical center-of-rotation is colinear with the midline of the fan beam.[1] This type of scanner is depicted in Fig. 1. In some situations it is not possible to align the midline with the mechanical center-of-rotation. Figure 2 depicts a scanner in which the center-of-rotation is perpendicularly displaced from the midline of the fan beam. In x-ray computed tomography (CT) a constant misalignment can occur when the x-ray source is mispositioned. A constant misalignment occurs in single photon emission computed tomography (SPECT) when the transaxial converging collimator that is used with a rotating gamma camera[2-4] is constructed with insufficient precision so that the focal point and the center-of-rotation do not project along the same ray. In CT and in SPECT it is also possible that the shift in the center-of-rotation varies with angular position because of mechanical instabilities in the gantry.

When data from a system with a constant misalignment are reconstructed without taking into consideration the shift in the midline, there will be a loss of resolution for 360° reconstructions.[5] For *half-scans*,[6] which are reconstructions from projections sampled over 180° plus the fan angle, structured artifacts will result.[7] When the shift in the center-of-rotation has an angular dependence, structured streaks will be present in reconstructed images. In the parallel beam case, it is possible to simply shift the projection prior to processing in order to account for the new center-of-rotation.[8] However, the fan beam case cannot be corrected by a simple shift, unless, of course, the fan beam projections are rebinned[9] into parallel beam projection sets. Rebinning, however is computational expensive and it degrades resolution.

Horn described reconstruction algorithms for arbitrary fan beam geometries for the case when the midline is colinear with the center-of-rotation.[10] Weinstein described a scanner where the focus-to-center distance is a function of rotation angle.[11] However, the results of these papers are not extendable to the misaligned scanner described here.

A reconstruction algorithm was derived assuming that

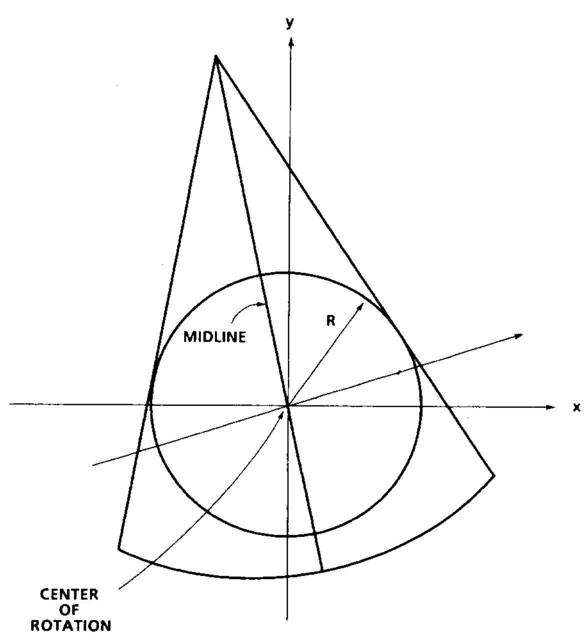

FIG. 1. Ideal fan beam geometry.

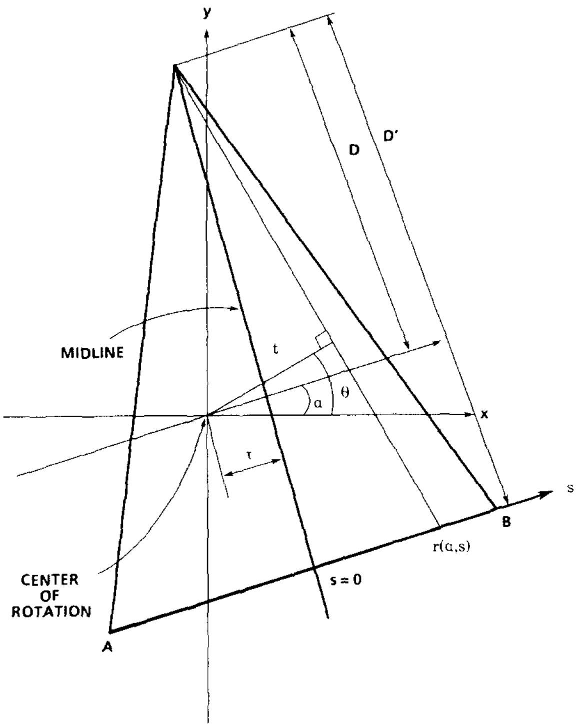

FIG. 2. Fan beam geometry with a displaced center-of-rotation.

the center-of-rotation was displaced from the midline of the fan beam by a distance that is constant for every angle.[12] In this paper we extend that work to derive a convolutional backprojection algorithm which corrects for the misalignment of the fan beam for the case of an angular-dependent shift. Simulations are given which show the effects of the shift and demonstrate the new algorithm's ability to correct for the misalignment. A method is also developed to estimate the parameters of the fan beam geometry including the angular-dependent shift in the center-of-rotation.

II. RECONSTRUCTION ALGORITHM DERIVATION

The function $f(x,y)$ is used to denote the cross section of an object to be reconstructed from its projections. The function $f(x,y)$ may be the distribution of the linear attenuation coefficients for x-ray CT or the radiopharmaceutical concentration for SPECT. It is assumed that the object is zero outside of a circle of radius R. A parallel projection of $f(x,y)$, $p_p(\theta,t)$, is the collection of line integrals through $f(x,y)$ along the paths given by

$$t = x\cos\theta + y\sin\theta, \tag{1}$$

for a fixed value of θ.

A polar coordinate version, $f(r,\phi)$, of the original function, $f(x,y)$, can be reconstructed from its projections using the following integral equation[13]:

$$f(r,\phi) = \int_0^{2\pi}\int_{-R}^{R} p_p(\theta,t)h\,[r\cos(\theta-\phi) - t\,]dt\,d\theta, \tag{2}$$

where $h(t)$ is given by

$$h(t) = \int_{-\infty}^{\infty} |\mu|/2\,\exp(2j\pi\mu t)d\mu. \tag{3}$$

Then $h(t)$ satisfies

$$h(at) = h(t)/a^2. \tag{4}$$

Consider the fan beam geometry depicted in Fig. 2 in which the midline is displaced from center-of-rotation by an angular-dependent value, $\tau(\alpha)$. It is assumed that $\tau(\alpha)$ and its derivative are continuous. A fan beam projection in this system is denoted by $p_f(\alpha,s)$. It can be seen from Fig. 2, that the fan beam and parallel projections are related as follows:

$$p_p(\theta,t) = p_f(\alpha,s), \tag{5}$$

for

$$t = [s + \tau(\alpha)]Z, \tag{6}$$

$$\theta = \alpha + \tan^{-1}(s/D), \tag{7}$$

where, for mathematical purposes, it has been assumed that the focus-to-detector distance, D', is equal to the focus-to-center distance, D, and

$$Z = D(s^2 + D^2)^{-1/2}. \tag{8}$$

Using Eqs. (6) and (7) the reconstruction formula given by Eq. (2) for the parallel projection case can be implemented in the fan beam (α,s) space. The components of the Jacobian for this transformation are

$$\frac{\partial t}{\partial s} = Z + [s + \tau(\alpha)]\,\frac{\partial Z}{\partial s} = [D^2 - \tau(\alpha)s]Z^3D^{-2}, \tag{9}$$

$$\frac{\partial t}{\partial \alpha} = \tau_\alpha Z, \tag{10}$$

$$\frac{\partial \theta}{\partial s} = D(s^2 + D^2)^{-1} = \frac{Z^2}{D}, \tag{11}$$

$$\frac{\partial \theta}{\partial \alpha} = 1, \tag{12}$$

where τ_α is the partial derivative of $\tau(\alpha)$ with respect to α.

The previous four equations can be combined to determine the Jacobian

$$J(s,\alpha) = [D^2 - \tau(\alpha)s - \tau_\alpha D\,]Z^3D^{-2}. \tag{13}$$

The transformation between the (t,θ) and (s,α) spaces has to be regular.[14] The transformation is regular if (1) it is continuous; (2) its partial derivatives are continuous; and (3) the Jacobian of the transformation is nonzero in the region of interest. Since the transformation given in Eqs. (6) and (7) is continuous and the partial derivatives of the transformation, Eqs. (9)–(12), are continuous, in order to prove regularity all we have to prove is that the Jacobian of the transformation does not go to zero. It can be shown that the Jacobian is greater than zero provided that the following condition is satisfied

$$D - \tau(\alpha)R\,/(D^2 - R^2)^{-1/2} > \tau_\alpha. \tag{14}$$

Using Eqs. (6), (7), and (13), we arrive at

$$f(r,\phi) = \int_0^{2\pi}\int_{-W}^{W} p_f(\alpha,s)\,[D^2 - \tau(\alpha)s - \tau_\alpha D\,]Z^3D^{-2} \times h\{r\cos[\tan^{-1}(s/D) + \alpha - \phi] - [s + \tau(\alpha)]Z\}ds\,d\alpha, \tag{15}$$

where W is the value of s for which $p_f(\alpha,s) = 0$ with $|s| > W$. The variable W is determined by letting $t = R$ in Eq. (6) and solving for s

$$W = \frac{DR\,[\tau^2(\alpha) + D^2 - R^2]^{1/2} - \tau(\alpha)D^2}{(D^2 - R^2)}. \quad (16)$$

The argument of the filter h in Eq. (15) can be written as

$$r\cos[\tan^{-1}(s/D) + \alpha - \phi] - [s + \tau(\alpha)]Z$$
$$= UZ(s' - s), \quad (17)$$

where

$$s' = [rD\cos(\alpha - \phi) - \tau(\alpha)D]/[r\sin(\alpha - \phi) + D], \quad (18)$$

$$U = [r\sin(\alpha - \phi) + D]/D. \quad (19)$$

Using Eqs. (17) and (4) we see that

$$h\{r\cos[\tan^{-1}(s/D) + \alpha - \phi] - [s + \tau(\alpha)]Z\}$$
$$= h(s' - s)/(UZ)^2. \quad (20)$$

The following is obtained when Eq. (20) is substituted into Eq. (15):

$$f(r,\phi) = \int_0^{2\pi} \frac{q(\alpha,s')}{U^2}\, d\alpha, \quad (21)$$

where

$$q(\alpha,s') = \int_{-W}^{W} p_f(\alpha,s)\, \frac{[D - \tau(\alpha)s/D - \tau_\alpha]}{(D^2 + s^2)^{1/2}}\, h(s' - s)ds. \quad (22)$$

Equation (21) represents a filtered backprojection algorithm for fan beam projections that have been collected with a shift in the center-of-rotation.

The following is a summary of the fan beam reconstruction algorithm:

(1) Multiply each fan beam projection, $p_f(\alpha,s)$, by $[D - \tau(\alpha)s/D - \tau_\alpha]/(D^2 + s^2)^{1/2}$.

(2) Convolve each weighted projection with $h(s)$.

(3) For each pixel in the reconstructed image and for each filtered projection, determine the value of the filtered projection at s' given in Eq. (18) and weight this value by U^{-2} given in Eq. (19). Add the weighted filtered projection value into the reconstructed image.

It should be noted that the adaptation of the analytic reconstruction algorithm to actual machine implementations can be done only with the introduction of approximations. The approximations deal with sampling considerations with regards to the kernel used to filter the weighted projection data and the conversion of the sampled filtered projections to continuously filtered projections.[15] In practice, the convolution indicated in Eq. (22) is performed using fast Fourier transform (FFT) operations incorporating the FFT of the filter, $h(t)$, and the FFT of the sampled projections, $p_f(\alpha,s)$. Because of noise and aliasing, the filter is rolled-off using a suitable window. Convolutional fan beam algorithms cannot be exactly derived if a window is used. However, because the window can be exactly incorporated into convolutional parallel beam reconstruction algorithms, it was assumed that the use of a window could also be used with the fan beam algorithm. The computer simulations that will be shown in the Sec. III verify that the use of a window will not adversely affect the quality of images. The reconstruction algorithm given in Eq. (21) can be combined with the material presented by Parker[6] to obtain a *half-scan* reconstruction algorithm for projections collected with an angular-dependent displaced center-of-rotation.

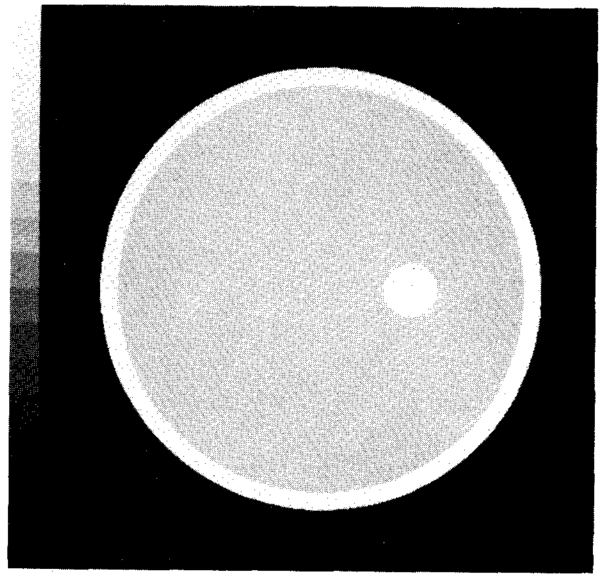

FIG. 3. Phantom used in the computer simulation studies.

III. COMPUTER SIMULATION RESULTS

A computer program was written to generate the x-ray line-integral data for the phantom shown in Fig. 3. A description of the ellipses used to construct the phantom can be found in Table I. A fan beam configuration (see Fig. 2) with a point source and a point detector was simulated with a focus-to-center distance D of 630 mm, a focus-to-detector distance D' of 1100 mm, and a detector spacing of 0.2 mm. Data were generated for 1000 projections with 768 samples/projection.

Figure 4 shows a normal 512×512 reconstruction, with a pixel size of 0.125 mm, of projection data generated without a shift in the center-of-rotation. The subtle streaks and other structured artifacts are due to aliasing and an insufficient number of views.

Line integral projection data were then generated for the case of an angular-dependent shift in the center-of-rotation. The shift, $\tau(\alpha)$, is given by

$$\tau(\alpha) = 3\sin(2\alpha)\ \text{mm}. \quad (23)$$

The reconstruction of the data, without correcting for the angular-dependent shift in the center-of-rotation, is shown in Fig. 5.

The reconstruction of the angular-dependent data, with correction for the shift, is shown in Fig. 6. The reconstruc-

TABLE I. Description of the ellipses used to construct the phantom shown in Fig. 3.

Ellipse	Origin X	Origin Y	Semimajor axes X	Semimajor axes Y	Rotation angle	Density
(1)	0 mm	0 mm	25 mm	25 mm	0°	1532 HU
(2)	0	0	23	23	0	−532
(3)	10	0	3	3	0	266

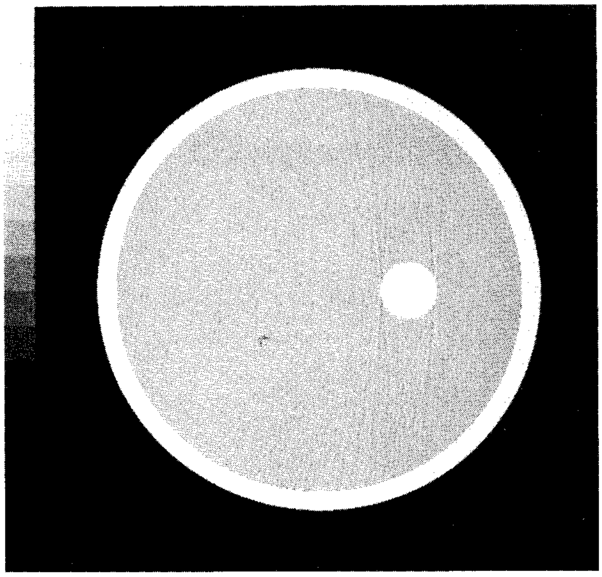

FIG. 4. Normal reconstruction of the phantom.

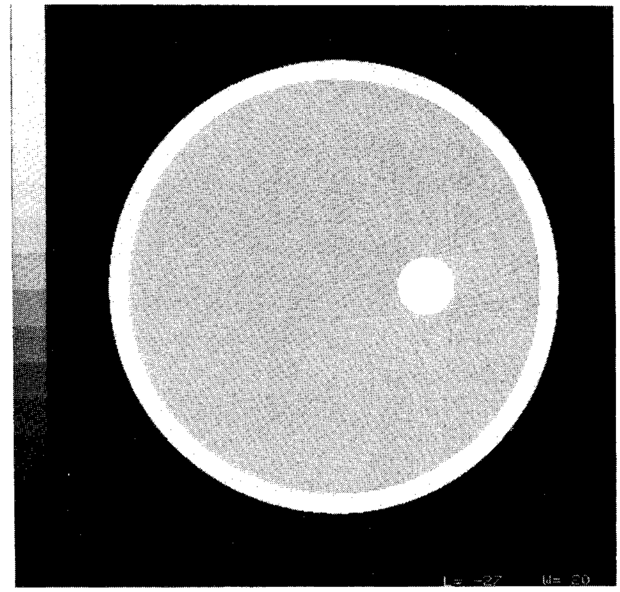

FIG. 6. Reconstruction with compensation for an angular-dependent shift.

tion algorithm used is given in Eqs. (21) and (22). It is clear that the new reconstruction algorithm corrects for a shift in the center-of-rotation.

IV. ESTIMATION OF FAN BEAM PARAMETERS

In parallel beam systems it is relatively easy to take calibrating measurements to determine the shift in the center-of-rotation. This is done by using a point source and taking complementary views 180° apart. The projection of the center-of-rotation onto the image plane is determined by summing the centroids of the projected point source and dividing by two. A small source of radioactivity is used in SPECT as the point source and a pin of highly attenuating material is used in x-ray CT.

In this section we show a method for measuring the parameters of the fan beam geometry shown in Fig. 7. The parameters are the angular-dependent displaced center-of-rotation $\tau(\alpha)$, the focus-to-center distance D, and the focus-to-detector distance D'. Mathematically we use

$$f(x,y) = \delta(x - x_0)\delta(y - y_0), \tag{24}$$

for a point source located at (x_0,y_0) to develop a relationship between the parameters of the fan beam geometry shown in Fig. 7 and something we can measure, namely, the centroids of the projected point source.

The fan beam projection operator[16] for the geometry in Fig. 7 is

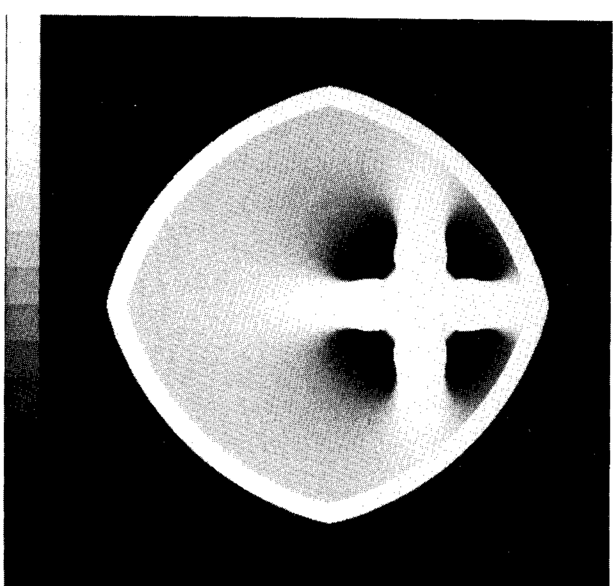

FIG. 5. Reconstruction without compensation for an angular-dependent shift.

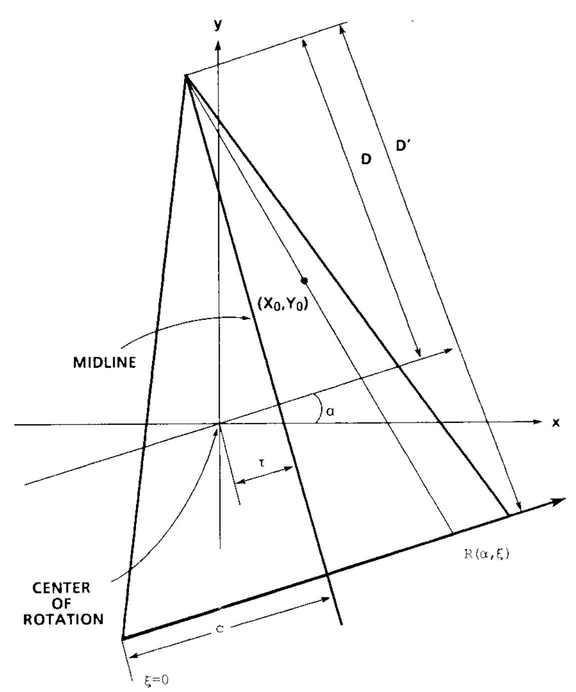

FIG. 7. The estimated parameters for the fan beam geometry.

$$R(\alpha,\xi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta\left(\frac{(\xi-c)(x\sin\alpha-y\cos\alpha+D)}{D'}-x\cos\alpha-y\sin\alpha+\tau(\alpha)\right)dx\,dy. \quad (25)$$

The projections of the point source are obtained by substituting Eq. (24) into Eq. (25):

$$R(\alpha,\xi)=\delta[(\xi-c)(x_0\sin\alpha-y_0\cos\alpha+D)/D' - x_0\cos\alpha-y_0\sin\alpha+\tau(\alpha)]. \quad (26)$$

For the angle α, the centroid of a projection is $\rho(\alpha)$ defined by

$$\rho(\alpha)=\int_{-\infty}^{\infty} R(\alpha,\xi)\xi\,d\xi \Big/ \int_{-\infty}^{\infty} R(\alpha,\xi)d\xi. \quad (27)$$

Substituting Eq. (26) into Eq. (27) and integrating we obtain

$$\rho(\alpha)=\frac{D'[x_0\cos\alpha+y_0\sin\alpha-\tau(\alpha)]}{(x_0\sin\alpha-y_0\cos\alpha+D)}+c. \quad (28)$$

Here, $\tau(\alpha)$ can be written in terms of its Fourier series:

$$\tau(\alpha)=\beta_0+\sum_{k=1}^{\infty}\beta_k\sin(k\alpha)+\eta_k\cos(k\alpha). \quad (29)$$

Assume that $\tau(\alpha)$ can be approximated by the DC term plus the first N terms of the Fourier series expansion, where N is a small number.

The result in Eq. (28) gives an expression for the projected centroid of a point source in terms of the fan beam parameters. This suggests a method to estimate the geometry of a fan beam system. In practice a point source is placed in the field of view of the scanner. Projections of the point source are collected and the centroid $\hat{\rho}_i$ is calculated for each angle α_i using Eq. (27). The parameters of the fan beam geometry can be estimated by minimizing the chi-square function:

$$\chi^2(x_0,y_0,D,D',\beta_0,\ldots,\beta_N,\eta_1,\ldots,\eta_N)=\sum_i[\hat{\rho}_i-\rho(\alpha_i)]^2, \quad (30)$$

where $\rho(\alpha_i)$ is given in Eq. (28). The process of minimizing Eq. (30) to determine estimates of the fan beam geometry is a nonlinear estimation problem which can be solved using the Marquardt algorithm.[17,18]

V. CONCLUSIONS

A method has been shown for determining and correcting for angular-dependent shifts in the center-of-rotation of a fan beam computed tomographic system. The projection data are preprocessed by multiplying by weighting factors which incorporate the displacement of the center-of-rotation. The modified projections are filtered and then backprojected correctly into a coordinate system whose center-of-rotation is displaced from the midline of the fan beam.

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[1] G. T. Herman and A. Naparstek, SIAM J. Appl. Math **33**, 511 (1977).
[2] R. J. Jaszczak, L. T. Chang, and P. H. Murphy, IEEE Trans. Nucl. Sci. **26**, 610 (1979).
[3] C. B. Lim, L. T. Chang, and R. J. Jaszczak, IEEE Trans. Nucl. Sci. **27**, 559 (1980).
[4] B. M. W. Tsui, G. T. Gullberg, E. R. Edgerton, D. R. Gilland, J. R. Perry, and W. H. McCartney, J. Nucl. Med. **27**, 810 (1986).
[5] M. Kijewski and P. Judy, Med. Phys. **10**, 169 (1983).
[6] D. L. Parker, Med. Phys. **9**, 254 (1982).
[7] L. A. Shepp, S. K. Hilal, and R. A. Schultz, Comput. Graphics Image Proc. **10**, 246 (1979).
[8] M. Dennis, R. Waggener, W. McDavid, W. Payne, and V. Sank, Opt. Eng. **16**, 6 (1977).
[9] P. Dreike and D. P. Boyd, Comput. Graphics Image Proc. **5**, 459 (1976).
[10] B. K. P. Horn, IEEE Proc. **67**, 1616 (1979).
[11] F. S. Weinstein, J. Opt. Soc. Am. **70**, 931 (1980).
[12] G. T. Gullberg, C. R. Crawford, and B. M. W. Tsui, IEEE Trans. Med. Imaging **5**, 23 (1986).
[13] A. Rosenfeld and A. C. Kak, *Digital Picture Processing*, 2nd ed. (Academic, New York, 1982), Chap. 8.
[14] D. Nahamoo, C. R. Crawford, and A. C. Kak, IEEE Trans. Biomed. Eng. **28**, 79 (1981).
[15] C. R. Crawford and A. C. Kak, Appl. Opt. **18**, 3704 (1979).
[16] G. T. Gullberg, Comput. Graphics Image Proc. **10**, 30 (1979).
[17] D. W. Marquardt, J. Soc. Ind. Appl. Math. **11**, 431 (1963).
[18] G. T. Gullberg, B. M. W. Tsui, C. R. Crawford, and E. R. Edgerton, Phys. Med. Biol. **32**, 1581 (1987).